Lena Pang

Dr. Nathan Cahill

MATH-421 Mathematical Modeling

6 November 2024

Module 4: Lotka-Volterra Internal Competition Model

Executive Summary

Overview

We derived the governing system for and created a model of the compartment interactions in a Lotka-Volterra predator-prey model that includes internal competition death rates.

Assumptions

To create our governing system of equations, we first start with our assumptions. Unlike the in-class exercise, we introduce internal competition death rates, where the death rate of a population is proportional to its own size. The complete assumptions are are as follows:

(a) The per capita prey birth rate remains constant.
(b) The per capita prey natural death rate remains constant.
(c) The per capita prey death rate due to internal competition depends linearly on the size of the prey population.
(d) The per capita prey death rate due to predation depends linearly on the predator population.
(e) The per capita predator birth rate depends linearly on the availability of prey.
(f) The per capita predator natural death rate remains constant.
(g) The per capita predator death rate due to internal competition depends linearly on the size of the predator population.

Governing System

For each of our assumptions, we can define a respective positive constant of proportionality (where all are proportional to the size of the population).

(a) Prey birth rate: α_1
(b) Prey natural death rate: α_2
(c) Prey competition death rate: $\beta x(t)$
(d) Prey predation death rate: $\gamma y(t)$
(e) Predator birth rate: $\delta x(t)$

(f) Predator natural death rate: σ

(g) Predator competition death rate: $\varepsilon y(t)$

For the sake of simplicity, we can define a net growth rate for the prey population, $\alpha = a_1 - \alpha_2$. We can use these to write a system of differential equations, where $x(t)$ is the size of the prey population at time t, and $y(t)$ is the size of the prey population at time t:

$$x'(t) = \alpha x(t) - \beta x(t)^2 - \gamma y(t)x(t)$$

$$y'(t) = \delta x(t)y(t) - \sigma y(t) - \varepsilon y(t)^2$$

Then we can simplify to create our standard-form governing system:

$$x'(t) = \alpha x(t)\left[1 - \frac{\beta}{\alpha}x(t) - \frac{\gamma}{\alpha}y(t)\right]$$

$$y'(t) = -\sigma y(t)\left[1 - \frac{\delta}{\sigma}x(t) + \frac{\varepsilon}{\sigma}y(t)\right]$$

Equilibrium Points

Let X, Y be the prey and predator population sizes at the equilibrium point(s). First, we look at the prey population. To have equilibrium in the prey population, we set the change in population size, $x'(t)$, to 0:

$$0 = \alpha X\left[1 - \frac{\beta}{\alpha}X - \frac{\gamma}{\alpha}Y\right]$$

So we get $\alpha X = 0$ or $1 - \frac{\beta}{\alpha}X - \frac{\gamma}{\alpha}Y = 0$. So we get three solutions, the first of which is trivial:

$$(X, Y) = (0, any) \qquad \text{A}$$

$$(X, Y) = (\frac{\alpha}{\beta}, 0) \qquad \text{B}$$

$$\beta X + \gamma Y = \alpha \qquad \text{C}$$

Then we can look at the predator population, also setting the change in size to 0:

$$0 = -\sigma Y\left[1 - \frac{\delta}{\sigma}X + \frac{\varepsilon}{\sigma}Y\right]$$

So we get $\sigma Y = 0$ or $1 - \frac{\delta}{\sigma}X + \frac{\varepsilon}{\sigma}Y = 0$. So we have three solutions, the first being trivial, and the second being realistically impossible (because all constants are defined as positive, it implies negative population):

$$(X, Y) = (any, 0) \qquad \text{I}$$

$$(X, Y) = (0, \frac{-\sigma}{\varepsilon}) \qquad \text{(invalid)}$$

$$\delta X - \varepsilon Y = \sigma \qquad \text{II}$$

So we get three overall equilibrium points:

$$(X, Y) = (0, 0) \qquad A \cap I$$

$$(X, Y) = (\tfrac{\alpha}{\beta}, 0) \qquad B \cap I$$

$$\beta X + \gamma Y = \alpha$$
$$\delta X - \varepsilon Y = \sigma \qquad C \cap II$$

We can rewrite the last system as the augmented matrix below and reduce:

$$\left[\begin{array}{cc|c} \beta & \gamma & \alpha \\ \delta & -\varepsilon & \sigma \end{array}\right]$$

$$\left[\begin{array}{cc|c} \beta & \gamma & \alpha \\ 0 & -\varepsilon - \gamma\frac{\delta}{\beta} & \sigma - \alpha\frac{\delta}{\beta} \end{array}\right]$$

$$\left[\begin{array}{cc|c} \beta & 0 & \alpha - \frac{\gamma\left(\sigma - \alpha\frac{\delta}{\beta}\right)}{-\varepsilon - \gamma\frac{\delta}{\beta}} \\ 0 & -\varepsilon - \gamma\frac{\delta}{\beta} & \sigma - \alpha\frac{\delta}{\beta} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{\alpha - \frac{\gamma\left(\sigma - \alpha\frac{\delta}{\beta}\right)}{-\varepsilon - \gamma\frac{\delta}{\beta}}}{\beta} \\ 0 & 1 & \frac{\sigma - \alpha\frac{\delta}{\beta}}{-\varepsilon - \gamma\frac{\delta}{\beta}} \end{array}\right]$$

When we simplify the fractions, we get the final equilibrium point:

$$(X, Y) = \left(\frac{\alpha\varepsilon + \sigma\gamma}{\beta\varepsilon + \delta\gamma}, \frac{\alpha\delta - \sigma\beta}{\varepsilon\beta + \gamma\delta}\right)$$

—

Equilibrium Volatility from the Jacobian

We set up the Jacobian for each equilibrium point to determine whether they are attractors or repellers. Keep in mind that all proportionality constants were defined as positive, so their negations are correspondingly negative.

We let the vector $\hat{z} = [x(t), y(t)]$.

Then we let the function $f(\hat{z})$ be our system:

$$f_1(\hat{z}) = \alpha z_1 - \beta z_1^2 - \gamma z_2 z_1$$

$$f_2(\hat{z}) = \delta z_1 z_2 - \sigma z_2 - \varepsilon z_2^2$$

Then the Jacobian is as follows:

$$J\left(f(\hat{z})\right) = \begin{bmatrix} \frac{\partial}{\partial z_1} f_1(\hat{z}) & \frac{\partial}{\partial z_2} f_1(\hat{z}) \\ \frac{\partial}{\partial z_1} f_2(\hat{z}) & \frac{\partial}{\partial z_2} f_2(\hat{z}) \end{bmatrix} = \begin{bmatrix} \alpha - 2\beta z_1 - \gamma z_2 & -\gamma z_1 \\ \delta z_2 & \delta z_1 - \sigma - 2\varepsilon z_2 \end{bmatrix}$$

Evaluating the Jacobian at each of our solutions, we get:

Solution 1: $(X, Y) = (0, 0)$

$$J\left(f(\hat{z})\right) = \begin{bmatrix} \alpha & 0 \\ 0 & -\sigma \end{bmatrix}$$

Because the matrix is diagonal, the eigenvalues are α and $-\sigma$. There is at least one positive eigenvalue, which means that this solution is unstable and therefore a repeller.

Solution 2: $(X, Y) = (\frac{\alpha}{\beta}, 0)$

$$J\left(f(\hat{z})\right) = \begin{bmatrix} \alpha - 2\beta\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \\ 0 & \delta\frac{\alpha}{\beta} - \sigma \end{bmatrix}$$

Because the matrix is upper triangular, the eigenvalues are the diagonal: $\alpha - 2\alpha$, $\delta\alpha/\beta - \sigma$. These values can be positive or negative based on particular parameter values, and the solution can therefore be stable or unstable.

Solution 3: $(X, Y) = \left(\frac{\alpha\varepsilon+\sigma\gamma}{\beta\varepsilon+\delta\gamma}, \frac{\alpha\delta-\sigma\beta}{\varepsilon\beta+\gamma\delta}\right)$

$$J\left(f(\hat{z})\right) = \begin{bmatrix} \alpha - 2\beta\frac{\alpha\varepsilon+\sigma\gamma}{\beta\varepsilon+\delta\gamma} - \gamma\frac{\alpha\delta-\sigma\beta}{\varepsilon\beta+\gamma\delta} & -\gamma\frac{\alpha\varepsilon+\sigma\gamma}{\beta\varepsilon+\delta\gamma} \\ \delta\frac{\alpha\delta-\sigma\beta}{\varepsilon\beta+\gamma\delta} & \delta\frac{\alpha\varepsilon+\sigma\gamma}{\beta\varepsilon+\delta\gamma} - \sigma - 2\varepsilon\frac{\alpha\delta-\sigma\beta}{\varepsilon\beta+\gamma\delta} \end{bmatrix}$$

Plugging into MATLAB, we get that the matrix eigenvalues are:

$$-\frac{\alpha\beta\varepsilon+\alpha\square\varepsilon-\beta\varepsilon\sigma+\beta\sigma\gamma-\sqrt{\alpha^2\beta^2\varepsilon^2-2\alpha^2\beta\square\varepsilon^2+\alpha^2\square^2\varepsilon^2-4\alpha^2\square^2\varepsilon\gamma+2\alpha\beta^2\varepsilon^2\sigma+2\alpha\beta^2\varepsilon\sigma\gamma-2\alpha\beta\square\varepsilon^2\sigma+2\alpha\beta\square\varepsilon\sigma\gamma-4\alpha\square^2\sigma\gamma^2+\beta^2\varepsilon^2\sigma^2+2\beta^2\varepsilon\sigma^2\gamma+\beta^2\sigma^2\gamma^2+4\beta\square\sigma^2\gamma^2}}{2(\beta\varepsilon+\square\gamma)}$$

$$-\frac{\alpha\beta\varepsilon+\alpha\delta\varepsilon-\beta\varepsilon\sigma+\beta\sigma\gamma+\sqrt{\alpha^2\beta^2\varepsilon^2-2\alpha^2\beta\delta\varepsilon^2+\alpha^2\delta^2\varepsilon^2-4\alpha^2\delta^2\varepsilon\gamma+2\alpha\beta^2\varepsilon^2\sigma+2\alpha\beta^2\varepsilon\sigma\gamma-2\alpha\beta\delta\varepsilon^2\sigma+2\alpha\beta\delta\varepsilon\sigma\gamma-4\alpha\delta^2\sigma\gamma^2+\beta^2\varepsilon^2\sigma^2+2\beta^2\varepsilon\sigma^2\gamma+\beta^2\sigma^2\gamma^2+4\beta\delta\sigma^2\gamma^2}}{2(\beta\varepsilon+\delta\gamma)}$$

We can rewrite, for simplicity, as $-\frac{a \pm \sqrt{b}}{2c}$. Depending on if a, b, and c are negative, we get different positive and negative eigenvalues. Thus this Jacobian, like the second solution, is inconclusive.

Modeling the Solutions

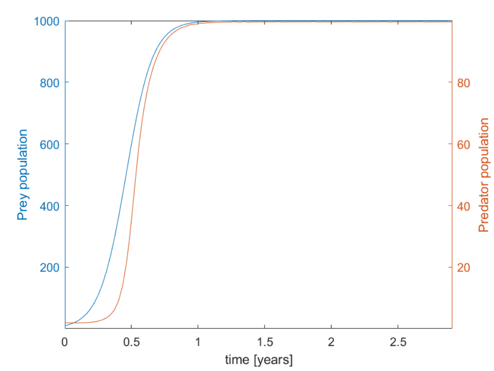
We implemented these solutions in MATLAB to model the timeline of each equilibrium point, using artificial tested values for the parameters.

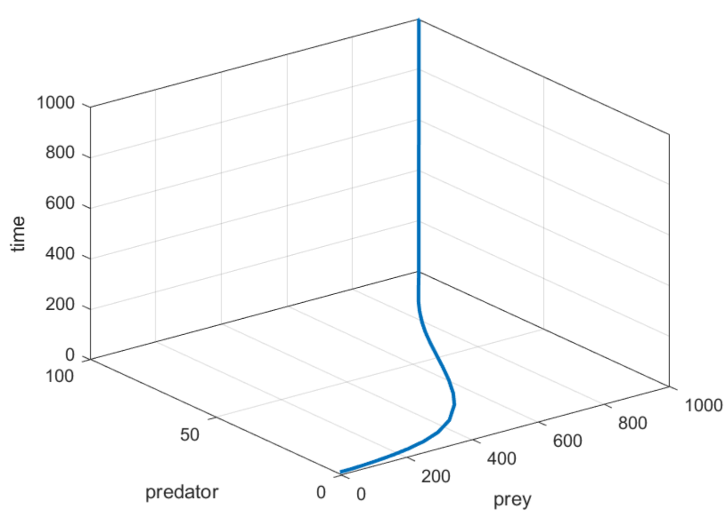
```
% parameters
alpha = 10;        % prey growth rate per capita
beta = .01;        % prey competition death rate per capita
gamma = .000001;   % prey predation death rate per capita per predator
delta = .05;       % predator birth rate per capita per prey
sigma = .2;        % predator natural death rate per capita
epsilon = .5;      % predator competition rate per capita

% system of equations
func = @(t,U) [alpha*U(1) - beta*(U(1)^2) - gamma*U(2)*U(1); ...
   delta*U(1)*U(2) - sigma*U(2) - epsilon*(U(2)^2)];

% solve Lotka-Volterra system numerically
[t,U] = ode45(func,[0,1000],U0);
```

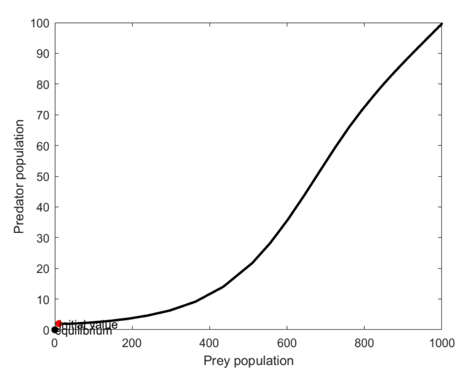
Solution 1: $(X, Y) = (0, 0)$

```
X_eq = 0; Y_eq = 0; U0 = [10, 2];
```

As we previously determined, this solution is a repeller. Starting with near-zero predator and prey populations causes dramatic increase and both before stabilizing.



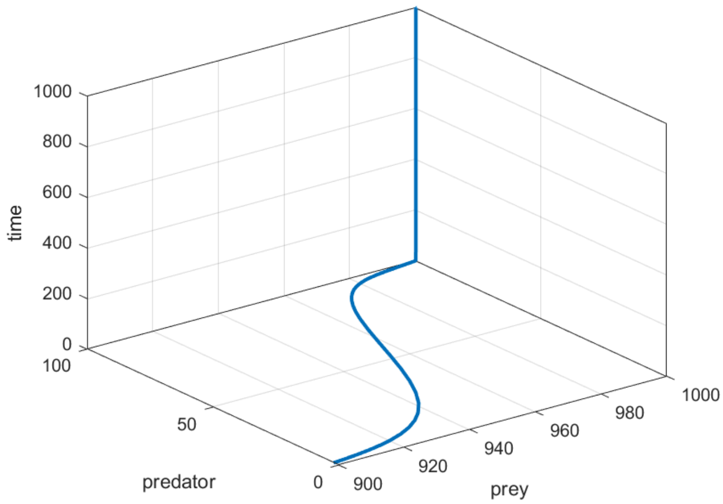


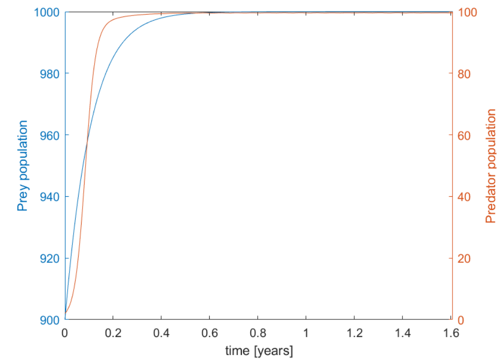


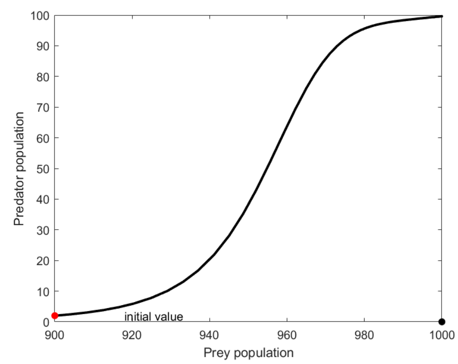
Solution 2: $(X, Y) = (\frac{\alpha}{\beta}, 0)$

```
X_eq = alpha/beta; Y_eq = 0; U0 = [0.9*X_eq, 2];
```

Starting near this equilibrium point, the prey and predator populations move away and later stabilize at Solution 3. So Solution 2 is a repeller.



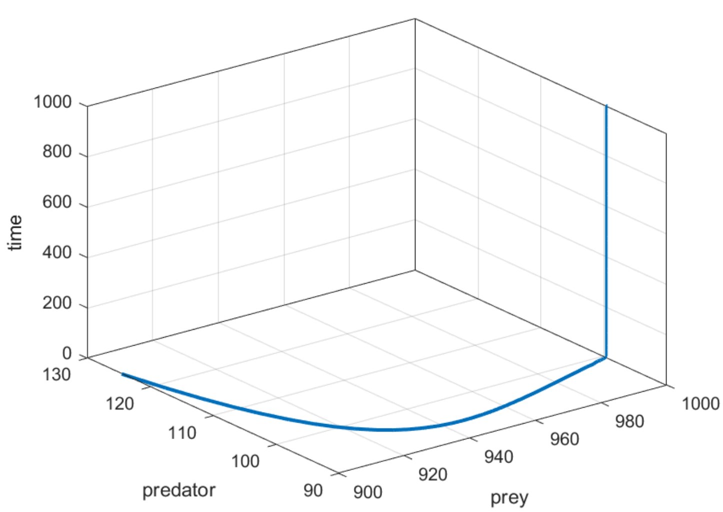


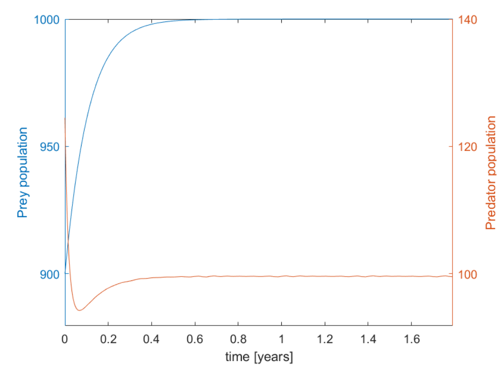


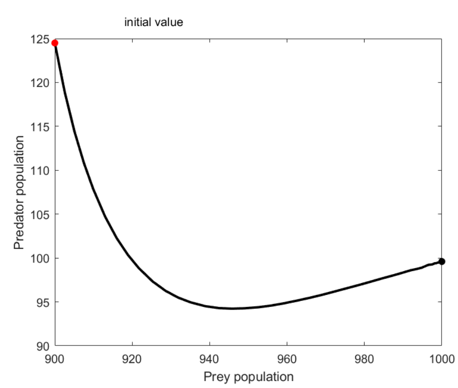
Solution 3: $(X, Y) = \left(\frac{\alpha\varepsilon+\sigma\gamma}{\beta\varepsilon+\delta\gamma}, \frac{\alpha\delta-\sigma\beta}{\varepsilon\beta+\gamma\delta}\right)$

```
X_eq = (alpha*epsilon + sigma*gamma) / (beta*epsilon +
delta*gamma); Y_eq = (alpha*delta - sigma*beta) /
(beta*epsilon + delta*gamma); U0 = [0.9*X_eq, 1.25*Y_eq];
```

Starting near this equilibrium point, the predator and prey populations both move towards the equilibrium point and stabilize. So Solution 3 is an attractor.







Conclusion

We created a system of equations to model the interactions between predator and prey populations including death rates due to internal competition. We determined that there are three equilibrium points $(X = prey,\ Y = predators)$ in this system: the repeller $(0, 0)$, the repeller $(\frac{\alpha}{\beta}, 0)$, and the attractor $\left(\frac{\alpha\varepsilon+\sigma\gamma}{\beta\varepsilon+\delta\gamma}, \frac{\alpha\delta-\sigma\beta}{\varepsilon\beta+\gamma\delta}\right)$. When graphed, starting at any nonzero populations leads to the attractor equilibrium point—as expected.

Unlike the model we created in class, this more complex model does have an attractor. As a result, the new timeline models are much less exciting than the one we saw in class: the attractor stabilizes both populations relatively quickly.